# SOME ADVANTAGES OF THE LOGARITHMIC SCALE IN STATISTICAL DIAGRAMS

Statisticians have long been aware that for some purposes of analysis and graphic representation the logarithmic scale has special advantages which the ordinary natural scale does not offer. But with that awareness they have mostly rested content. Comparatively few have worked with the logarithmic method and really come to know it in its applications. The logarithmic scale, it has been assumed, would be unintelligible save to experts. Because of its unfamiliarity it has been neglected; through neglect it remains unfamiliar, and thus a serviceable statistical tool lies virtually idle.

It is the thesis of this article that the logarithmic scale is too useful as a statistical auxiliary to be disregarded simply because it is not yet generally understood; and, further, that the best way to secure for it a more general understanding and appreciation is to use it intelligently at every appropriate opportunity. This is not by any means to say that the logarithmic scale should prevailingly replace the natural scale in ordinary graphic work. Often, however, where only one method is to be followed, the logarithmic construction is clearly superior. Even if such occasions are as yet comparatively infrequent, it is still true that figures drawn to the logarithmic scale will usually afford an informing contrast and supplement to diagrams in the more conventional form.

For demonstrating the characteristics of logarithmic diagrams such a contrast is quite the most promising plan. Mr. Bowley, in his brief but excellent discussion of logarithmic diagrams,[1] has recommended this procedure. "It would be useful," he remarks, "to offer several diagrams on both scales; for in many series of figures the differences exhibited by the two methods are very instructive." Mr. Bowley felt himself constrained by the limits of space to forego acting upon his own proposal, but the suggestion was good. In the following pages space is deliberately devoted to

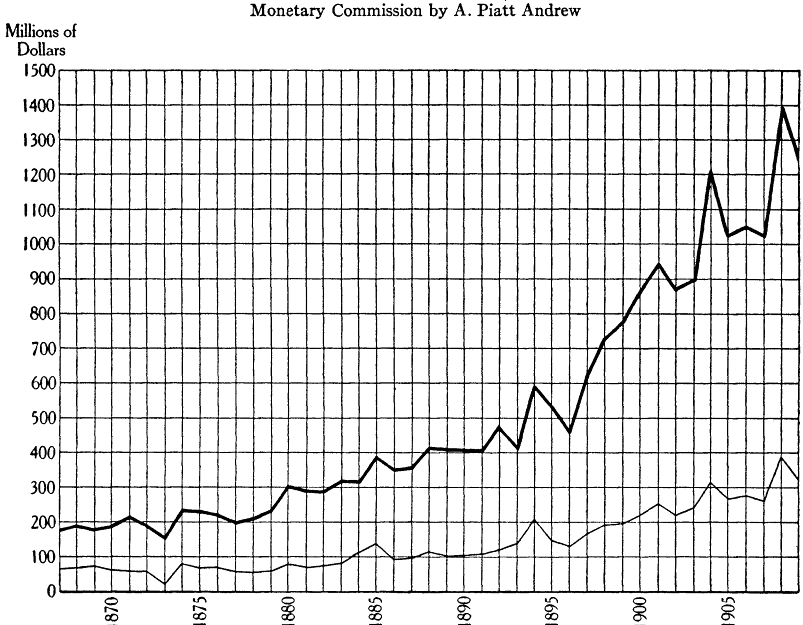
[1] *Elements of Statistics*, pp. 188 ff.

a comparative display of diagrams of the two kinds. The primary intention is to exemplify, not to innovate. There is no pretense of any important originality. The purpose of the article is simply

DIAGRAM I.—NET DEPOSITS (HEAVY LINE) AND RESERVES (LIGHT LINE) OF THE CLEARING-HOUSE BANKS OF NEW YORK CITY, ACCORDING TO THE 41ST WEEKLY REPORT (EARLY OCTOBER) IN EACH YEAR, 1867–1909

*Natural Scale*

Data (except for the year 1888) from *Statistics for the United States, 1867–1909*, compiled for the National Monetary Commission by A. Piatt Andrew




to make logarithmic diagrams a little less unfamiliar, and to let them speak for themselves on the basis of better acquaintance.[1]

The graphic method in statistics is primarily a device for presenting vividly the significant relations of phenomena. Each slope of a curve in an ordinary two-dimension statistical diagram is the

[1] This article was substantially completed some months before the appearance (in the *Publications of the American Statistical Association*, June, 1917) of Professor Irving Fisher's ingenious demonstrations of the "Ratio Chart," which it unfortunately overlaps. It had itself been proposed as a contribution to the *Publications* two years ago; but at that time the editor of the *Publications* was not prepared to reproduce the requisite number of diagrams.

visible expression of some relationship. If the purpose of a particular statistical presentation is simply an accurate recording of separate details, a diagram is, of course, a poor substitute for plain numerical statements; but when the relative aspects of the data are to be emphasized the diagram comes into its own.

And yet, even within this sphere of its special excellence, graphic representation, in terms of the common, natural scale of uniform intervals, has very real limitations. Too frequently, though the problem is simple and the diagram is well done, the eye will fail to detect the precise nature of the relationship which the statistician seeks to present.

Some of the shortcomings of natural-scale representation are fairly illustrated by Diagram I. The upper and lower curves[1] of this figure show, respectively, the net deposits and the reserves of the New York Clearing-House banks in early October of each year from 1867 to 1909, inclusive. From the diagram in this form certain facts are indeed sufficiently clear. Both deposits and reserves increased markedly during the period under review. The increase of each, though on the whole progressive, has been subject to appreciable fluctuations; and the fluctuations of one curve are associated with synchronous and apparently similar fluctuations of the other.

[1] The term "curve," it must be noted, is used here in a loose sense. The data upon which Diagram I is based define only the points where the so-called curves cut the successive vertical ordinates. The straight lines connecting these points are quite arbitrary and do not at all necessarily represent actual intervening values of the phenomena plotted. They serve merely to link the given points together in such a way as roughly to suggest the general trend of increase or decrease during the interval between the recorded observations.

In some instances such arbitrary straight lines closely approximate the curve which would result if observations were made and plotted at very short intervals. This is probably the case in Diagram V. But in Diagram I the straight-line method of construction ignores the existence of a more or less characteristic annual cycle in the movement of bank deposits and reserves. In such cases as Diagram XV the method is almost wholly anomalous, since there the total tin production of an entire year is plotted at each ordinate, and any curve between the ordinates is therefore imaginary.

The construction of diagrams like Diagram I or Diagram XV is obviously inexact; but it follows a statistical tradition which unfortunately is almost rigidly established. From this tradition it has not seemed expedient to depart in the present article, lest the raising of a secondary issue should distract attention from the primary theme. It may be remarked, however, that the arbitrary straight line between two points has not in strictness the same meaning in a logarithmic diagram that it has in a diagram drawn to natural scale.

The amount of deposits or of reserve in the early days of any particular October may be estimated by consulting the scale at the side of the diagram. The amount of increase or decrease of either item during a given year or term of years is not difficult to determine approximately. All this information, then, the ordinary scale gives adequately. Some of it would be less satisfactorily given by any other scale. But if we press our inquiries further and ask, on the basis of these early October statements, whether, for example, the expansion of deposits was *relatively* greater in the year after the crisis of 1907 than in the year after the crisis of 1873, or whether the contraction of deposits was relatively greater before 1873 than before 1896; if we try to compare the percentages of reserve held in the years before 1870 with the corresponding figures since 1895; or if we wish to know specifically what was the percentage of reserve in early October of 1905, deficiencies of the natural scale are revealed. None of these questions, which concern relations rather than detached facts, is satisfactorily answered by the diagram. If answers are forthcoming at all, it is only because, through the scales, one may roughly and inconveniently recover the numerical data from which the diagram was made. This, however, could have been more easily accomplished by ignoring the diagram altogether and consulting its data in the form of a table.

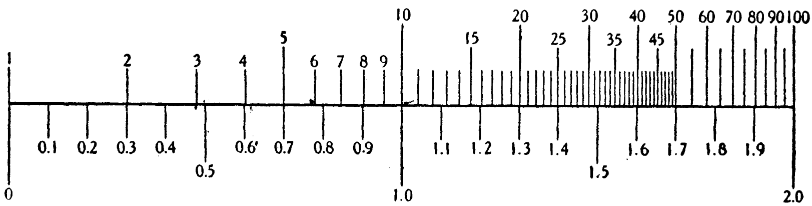
It is practicable, of course, to contrive a diagram, drawn to a natural scale, with the special purpose of bringing out some one fact or relation which in Diagram I has remained obscure. Thus the percentage of reserve of the New York banks could be plotted, year after year, as a separate curve. This curve, however, would in turn fail to show the absolute amounts of reserves and deposits. The difficulty is to devise a form of representation which shall show, directly and graphically, both relative and absolute magnitudes. A complete solution of this problem is hardly attainable, but logarithmic diagrams in certain cases go far toward meeting the want where the relative aspects of the phenomena are primarily to be emphasized.

The logarithmic scale may indeed be described as a scale of ratios. On it absolute distances measure relative magnitudes. The numbers which occur at equal intervals along a logarithmic

scale thus form not an arithmetic but a geometric progression; and consequently the same proportionate relation exists between any two numbers a given distance apart on a given logarithmic scale, regardless of their absolute magnitudes and regardless of their absolute differences. Conversely, the numbers 2 and 4 on a logarithmic scale are separated by the same distance as the numbers 500,000 and 1,000,000, for the simple and decisive reason that the larger number of each pair is double the smaller number.

The mathematical principle of the scale is suggested by Diagram II. Here the graduations above the horizontal line mark off the intervals of a logarithmic scale from 1 to 100. The feature of this scale which at once strikes the eye rather bewilderingly is that the interval between successive numbers is not constant, but

DIAGRAM II.—THE LOGARITHMIC SCALE FROM 1 TO 100




progressively narrows as the numbers grow larger. Closer scrutiny reveals the more significant and clarifying fact that the interval is constant between numbers which bear to each other a given ratio. Thus 1, 2, 4, 8, 16, 32 stand at equal distances apart; as do 1, 3, 9, 27, or 1, 5, 25, or 1, 10, 100. The uniform interval which separates the numbers of this last-named series—successive powers of 10—has been taken as the unit upon which is based the ordinary scale below the horizontal in the diagram. If, now, any number on the upper scale be regarded as a power of 10, it will be found that the corresponding reading of the lower scale gives the index of that power. This relation holds invariably; for not only do we find 10 (i.e., $10^1$) opposite 1, 100 (i.e., $10^2$) opposite 2, and 1 (i.e., $10^0$) opposite 0, but the square root of 10 (i.e., $10^{\frac{1}{2}}$, or 3.1623) is opposite 0.5; the square root of 1000 (i.e., $10^{\frac{3}{2}}$, or 31.623) is opposite 1.5—and so on indefinitely, whatever the index of the power. In fact,

the number at any point of the lower scale is the common logarithm of the number at the same point of the upper scale.[1]

If, now, it is desired to use the logarithmic scale in the construction of a statistical diagram, we may proceed in either of two ways. We may reduce the data to logarithmic terms, and then, using an ordinary natural scale, plot the logarithms of the given quantities instead of the quantities themselves. Or if we have at our disposal co-ordinate paper ruled at logarithmic intervals, like the intervals of the upper scale of Diagram II, we may work directly, without any reduction of the data, locating the points of the diagram quite mechanically by the graduations of the paper, and relying upon these graduations for the logarithmic character of the result.[2] The

[1] The system of logarithms which is in ordinary use expresses any given number as a certain power of 10. The logarithm of the given number indicates what power of 10 that number is. Thus the logarithm of 10 is 1; the logarithm of 100—i.e., of $10\times 10$, or $10^2$—is 2; the logarithm of 1000, or $10^3$, is 3, and so on. A logarithm is in fact an exponent—the index of a power—and the derivation and uses of logarithms consequently follow the algebraic rules of exponents. In the case of a number which is not an even power of 10 it is possible to compute the logarithm in the form of a fractional exponent. For example, as the text implies, the logarithm of 31.623, the square root of 1000—i.e., $\sqrt{10^3}$ or $10^{\frac{3}{2}}$—is 1.50. By extending the principle of fractional exponents the logarithm of any assignable number may be approximately expressed.

The peculiar advantage of the logarithmic scale in statistical work is a consequence of the elementary logarithmic principle that the difference between the logarithms of two numbers is the logarithm of the ratio of the one number to the other. That is,

$$\log a - \log b = \log \frac{a}{b}.$$

Hence, whenever the ratio between two numbers $a$ and $b$ is the same as the ratio between two other numbers $p$ and $q$, so that $\frac{a}{b}=\frac{p}{q}$, and $\log \frac{a}{b} = \log \frac{p}{q}$, it will follow that $\log a - \log b = \log p - \log q$. Plotted to a given natural scale, $\log a$ and $\log b$ would thus differ by the same interval as $\log p$ and $\log q$—the equality of these differences indicating the equality of the ratios $\frac{a}{b}$ and $\frac{p}{q}$. The device of plotting statistical quantities in terms of their logarithms is, then, simply an exploiting of the general principle that the absolute difference between two logarithms is a measure of the relative difference of the numbers to which they correspond.

[2] For an instance of tolerably elaborate logarithmic graduation see Diagram XII, on p. 828.

Suitable logarithmic co-ordinate paper is unfortunately not easily secured. Such logarithmic papers as are available are mostly designed for the use of engineers and are for one reason or another hardly satisfactory for general statistical use. The logarithmic diagrams which accompany this article have in the main been drawn on paper specially engraved for the statistical laboratory of the University of Chicago.

two methods are entirely equivalent, as should be evident from Diagram II. Indeed it is often convenient to regard a diagram as constructed by both methods, and to supply for its more complete explanation a logarithmic scale of the natural numbers on one side, and a natural scale of their logarithms on the other.[1]

Before attempting a logarithmic presentation of the bank data of Diagram I, it will be well to consider, in artificially simplified cases, certain general properties of logarithmic diagrams which furnish the key to their interpretation.

Let us take for our first illustration the arbitrary example of Diagram III. Here an assumed phenomenon, which has a magnitude of 1 when it is first observed, increases to 5 in the course of a year and then, in the second year, falls off to $2\frac{1}{2}$. In the third year it again increases fivefold, to $12\frac{1}{2}$. In the fourth year it again declines by half, to $6\frac{1}{4}$. Thus alternately quintupled and cut in two, the phenomenon grows by perfectly regular oscillations. Diagram III, which is drawn to an ordinary natural scale, shows vividly the accelerated character of this increase, stated in absolute numbers; but precisely because it is a natural-scale diagram it fails to show at all obviously that the rate of relative rise and fall is the same for all the oscillations. The earlier waves of the curve, which are absolutely small, are made to seem in all respects comparatively insignificant.

Strikingly different is the effect of Diagram IV, in which the data of Diagram III are plotted to a logarithmic scale. Absolute magnitudes here can be determined only from the numbers of the scale: the graphic evidence of the diagram establishes the identity of the relative changes, step by step, for the whole serrate curve. Every ascent has the same vertical rise. That is, the indicated percentages of increase are uniform. Each decline has the same drop: the percentage of decrease shown by each is the same. This equal relative significance of equal absolute distances is the essential characteristic of the logarithmic scale.

Certain fairly obvious but important corollaries follow from this fundamental principle. Since the upstrokes of the curve in

[1] For an example of this treatment see Diagram IV on p. 812.

DIAGRAM III.—ARBITRARY EXAMPLE OF A PHENOMENON INCREASING BY EQUAL RELATIVE OSCILLATIONS

*Natural Scale*

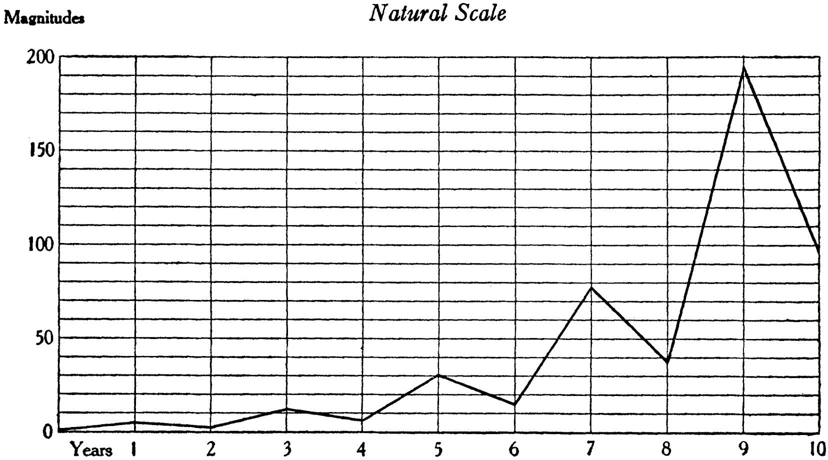



DIAGRAM IV.—ARBITRARY EXAMPLE OF A PHENOMENON INCREASING BY EQUAL RELATIVE OSCILLATIONS

*Logarithmic Vertical Scale*

Data of Diagram III

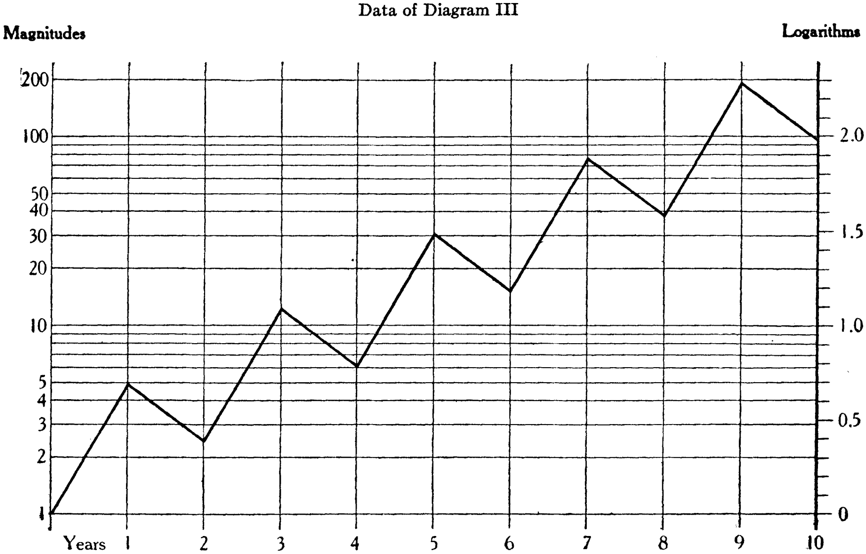


Diagram IV are all straight lines rising by the same amount, and since each rise, occurring in the same period of time, is allotted in the diagram the same horizontal distance, it follows that the slope of the several upstrokes is the same. The downstrokes are similarly all of the same slope. Quite generally, where a curve is drawn to a logarithmic vertical scale and a natural horizontal scale,[1] equal slopes indicate equal rates of relative change. By extension of this rule it will be seen that a constant rate of increase is represented in a logarithmic curve by a constant slope—i.e., by a straight line; and that wherever in such a logarithmic diagram two curves run parallel, in the sense that the vertical distance between them remains unaltered,[2] the phenomena which they respectively represent maintain to each other a constant ratio, inasmuch as any change of the one is evidently coincident with a change of the other to the same relative extent.

These generalizations may be simply illustrated by the examples which follow.

In Diagram V, drawn to natural scale, the continuous curve traces the growth of the population of the United States, according to the decennial enumerations of the United States

[1] The rules here stated also apply, *mutatis mutandis*, to diagrams in which the horizontal scale is logarithmic and the vertical scale an ordinary scale. Such constructions are, however, unusual. Cf. below, pp. 833 ff.

[2] This qualification is necessary when the curves in question are not straight lines. Similar arcs of two concentric circles, for example, although in a sense parallel (i.e., equidistant at all points in terms of the normals to the curves), would not, in an ordinary logarithmic diagram, imply a constant ratio between the quantities plotted. On the other hand, curves which are parallel in the sense of the text, and which do therefore indicate a constant ratio between two variables, may not be parallel in appearance. This difficulty, illustrated by the accompanying figure, suggests that the effectiveness of logarithmic diagrams, when they are designed to test the proportionality of the data plotted, is much impaired if the contrasted curves are far from straight.

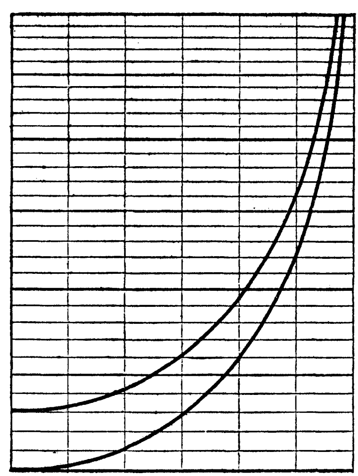
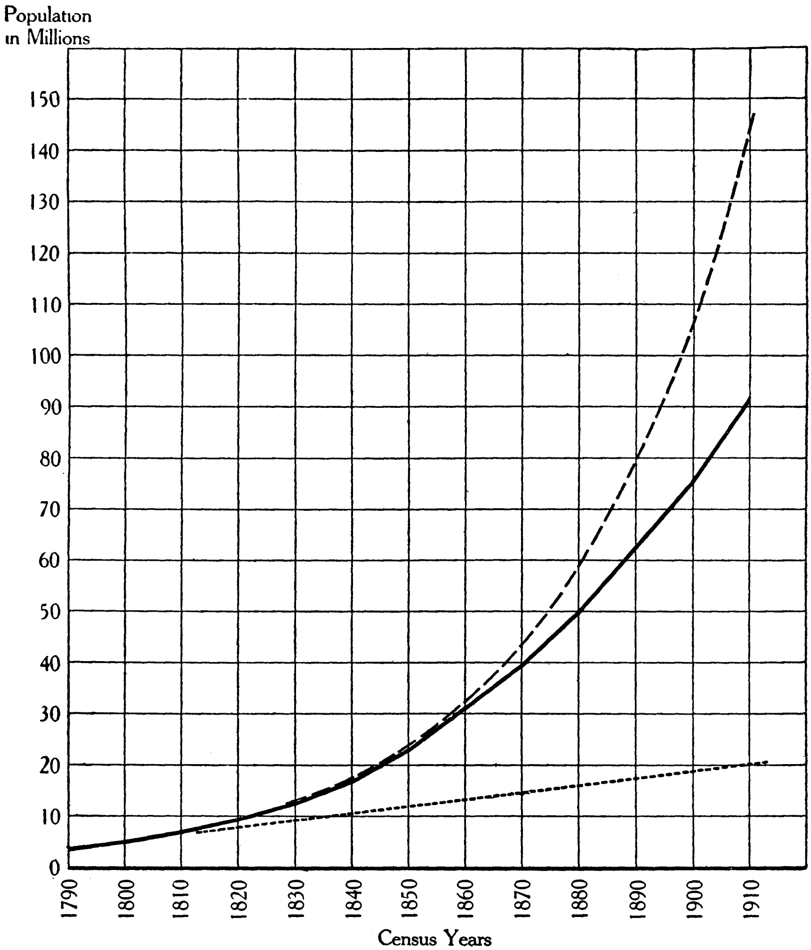


DIAGRAM V.—GROWTH OF THE POPULATION OF THE UNITED STATES, 1790–1910

The continuous line shows the actual increase according to the census returns. The broken and dotted lines show the growth which would have taken place if relative and absolute increase, respectively, had continued at the rate of the first decade.

*Natural Scale*

Data from 13th Census of the United States, I, 24. The corrected estimate for 1870 has been taken instead of the original enumeration

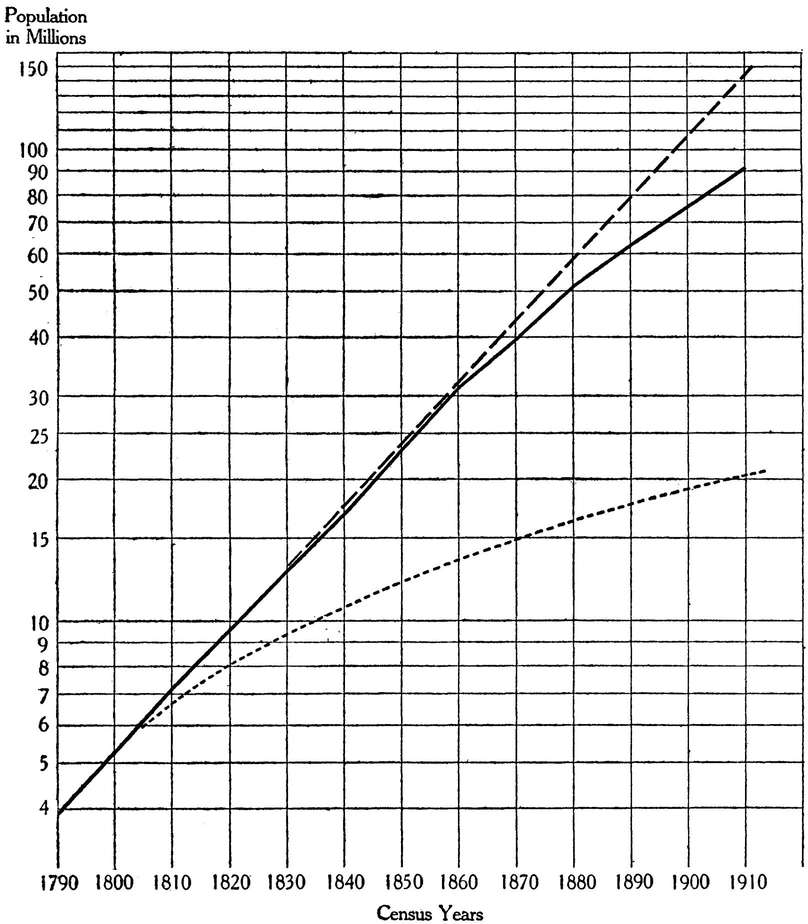


Census,[1] from 1790 to 1910, inclusive. The broken line, uppermost in the diagram, shows what the growth of population would have been

DIAGRAM VI.—GROWTH OF THE POPULATION OF THE UNITED STATES, 1790–1910

*Logarithmic Vertical Scale*

Data and explanations as in Diagram V




[1] For the population in 1870 the revised estimate of 39.8 millions has been accepted in preference to the original enumeration of 38.6 millions. (Cf. *Eleventh Census*, "Population," Part I, xi, xii.) No attempt has been made to indicate in the diagram the fact that not all intercensal intervals have been of precisely ten years.

if the rate of relative increase observed between 1790 and 1800—35.1 per cent for the decade—had persisted without change since that time. The dotted line at the bottom of the figure shows what the growth would have been if the absolute increase of population in each decade since 1800 had been the same as the increase—1,379,269 persons—from 1790 to 1800. In other words, these two additional curves represent respectively geometric and arithmetic progressions based on the observed increase in the first intercensal period. It is to be noted that in a natural-scale construction the curve of arithmetic progression is a straight line.

In Diagram VI, drawn to a logarithmic scale, the continuous line, the broken line, and the dotted line represent each the same data as in Diagram V. But here the character of the curves is significantly different. The dotted arithmetic-progression curve, recording a constantly diminishing ratio of increase, falls away in this figure more and more toward the horizontal. And here it is the geometric progression which appears as a straight line, its constant slope denoting a constant rate of increase—i.e., the same relative increase in every equal period of time.

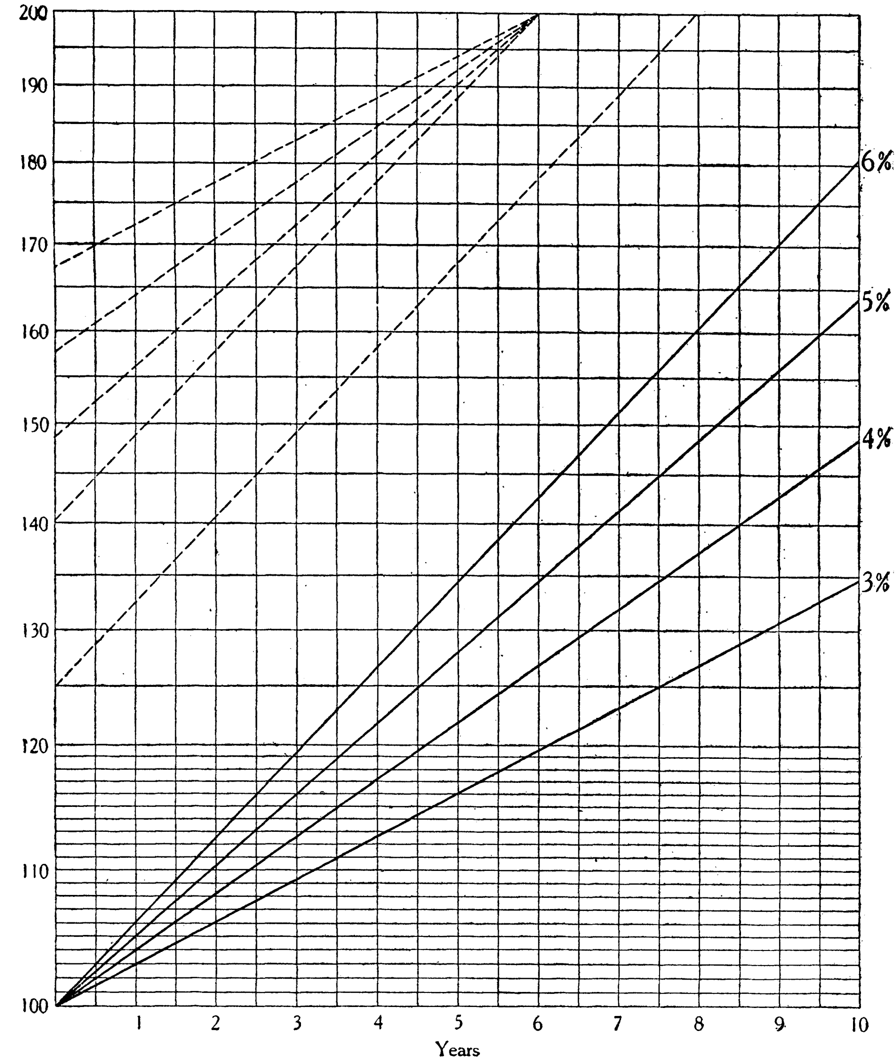
The growth of funds invested at compound interest affords another instance of geometric increase and therefore another example of a straight-line curve if a diagram is drawn to a logarithmic scale. The slope of the curve here depends upon the rate of interest and the interval between dates at which the interest is regularly compounded; but for a given rate and interval it is fixed and constant. Hence a logarithmic chart equivalent to a compound-interest table may very readily be constructed. Diagram VII is such a chart. In it a single straight line suffices to indicate the amount to which an initial sum of $100, compounded semi-annually at a given rate, will have increased on any compounding date included in the diagram.[1] The 4 per cent line is steeper than the 3 per cent line; the 5 and 6 per cent lines are successively

[1] The period of time covered by such a chart is of course in principle unlimited, for the lines will continue with their same specific slopes however far the diagram may be extended.

steeper still; but all are straight, and for each, when the scales of the diagram are once determined, the slope is fixed and characteristic.

Diagram VII.—Compound-Interest Chart (Semiannual Compounding)

*Logarithmic Vertical Scale*



The same diagram serves also to illustrate another property of logarithmic diagrams that has already been mentioned. The broken line across the middle of the figure has been drawn to show the increase of $125, compounded semiannually at 6 per cent. It is at once apparent that this line parallels the continuous line of the increase of $100 at the same rate. The reason for the parallelism is tolerably patent. Each of the sums, $100 and $125, increases every six months by 3 per cent of its accumulated amount. That is, each sum is semiannually multiplied by 1.03. In the diagram, therefore, each of the two lines must rise, from one ordinate to the next, by the fixed vertical distance which, on the logarithmic scale, corresponds to the ratio 1.03:1.00. This, of course, insures that both rise alike. Or it may rather be argued that since original sums in the proportion of 1.25 to 1 are here assumed to be compounded at the same rate and the same interval, the cumulative results will be at any subsequent time in the same proportion of 1.25 to 1. The vertical distance between the two curves on any ordinate must therefore express the ratio 1.25:1.00, and hence, since a given ratio always corresponds to the same absolute interval on a logarithmic scale, the curves must be always at the same distance apart and therefore parallel. It follows that if a point be taken on the initial ordinate of this diagram, opposite the value $125 of the vertical scale, the straight line drawn through that point parallel to the original 6 per cent curve will represent the compound increase of $125 at 6 per cent. Similarly, to find the increase of any capital sum at any rate of compound interest, one has only to draw a straight line starting at the height which denotes the given sum and running parallel to a standard curve for the given rate of interest. In Diagram VII this principle has a somewhat different application. Through the point representing a sum of $200 at the end of 6 years have been drawn broken lines parallel to the standard curves showing respectively 3 per cent, 4 per cent, 5 per cent, and 6 per cent increase. These several broken lines cut the initial ordinate at heights which, read in terms of the vertical scale, show what amount of money, compounded semiannually at each respective rate of interest, would amount to $200 after 6 years.

If logarithmic diagrams thus simply indicate the results of compound interest, they will with equal simplicity lend themselves to the elucidation of problems of depreciation when it is assumed that annual depreciation is a fixed percentage of the residual value of the asset at the beginning of the year. Depreciation thus defined is "straight-line" depreciation according to the logarithmic scale, just as depreciation by equal annual deductions is "straight-line" depreciation in the terminology of natural-scale graphics.

In principle the logarithmic method would serve as well to interpret observed facts of depreciation as to apply the fixed-percentage hypothesis. If the facts could be plotted in logarithmic charts the slopes of the curves would disclose the real rates of depreciation. Since, however, the treatment of depreciation in accounting is usually based on more or less arbitrary assumptions rather than on actual appraisals, it will probably be more interesting in the present connection to consider a somewhat analogous case where data are available.

Diagrams VIII and IX present in graphic form data from the "Life-Table for Native White Males in the Original Registration States," computed under the direction of Professor Glover by the Bureau of the Census. The height of the curve at any ordinate in these figures shows how many of 100,000 born survive to the specified age. The drop of the curve from one age to the next thus reveals the mortality between the two ages. But the sense in which these slopes indicate mortality is quite different for the two diagrams. In the natural-scale diagram (Diagram VIII) the descent of the curve expresses the number of deaths in a year among the survivors to a given age. This is not the usual way of stating death-rates; nor is it a convenient method, since the absolute number of deaths is a joint resultant of two factors which might better be considered separately—the probability of death at the specified age, and the number of persons at that age and subject to that hazard. We are ordinarily more concerned with the probability alone, or, which is much the same thing, with the proportion of those persons of given age who die in the course of a year. Precisely this relative mortality rate determines the slope of the curve in the logarithmic figure (Diagram IX), for here, as always, a given

DIAGRAM VIII.—MORTALITY AND SURVIVAL OF NATIVE WHITE MALES IN THE ORIGINAL REGISTRATION STATES OF THE UNITED STATES, 1910

*Natural Scale*

Data from *United States Life Tables, 1910*, pp. 30–31

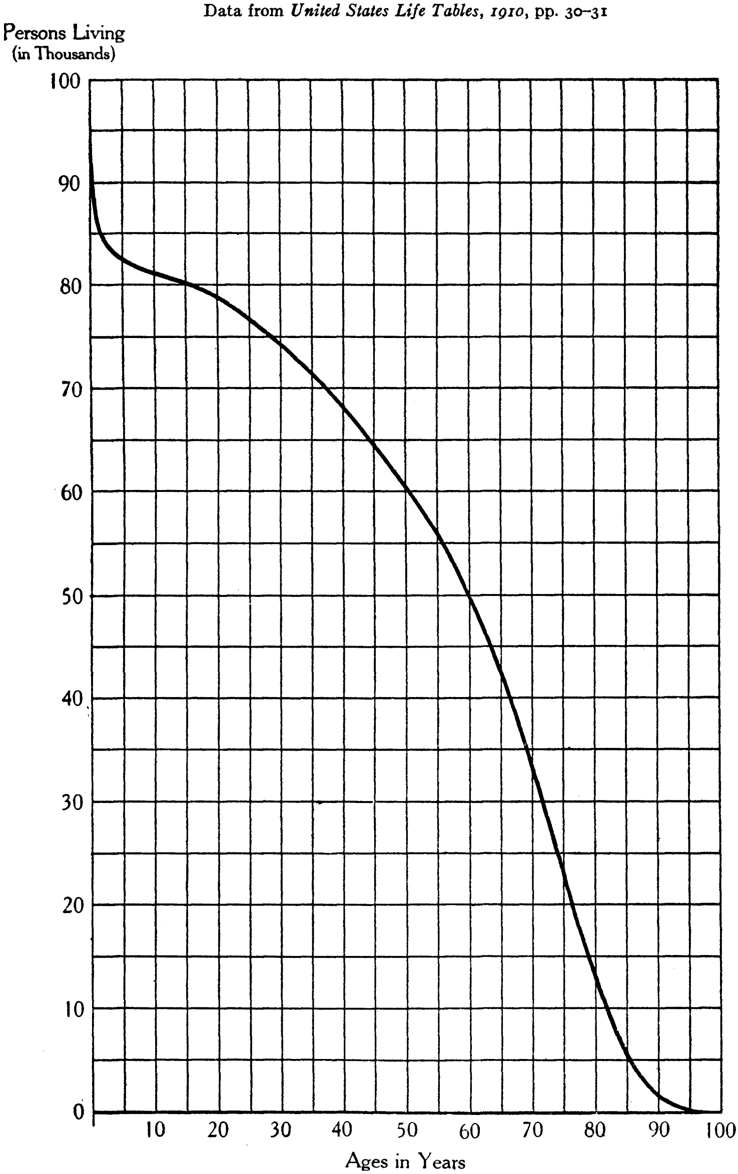
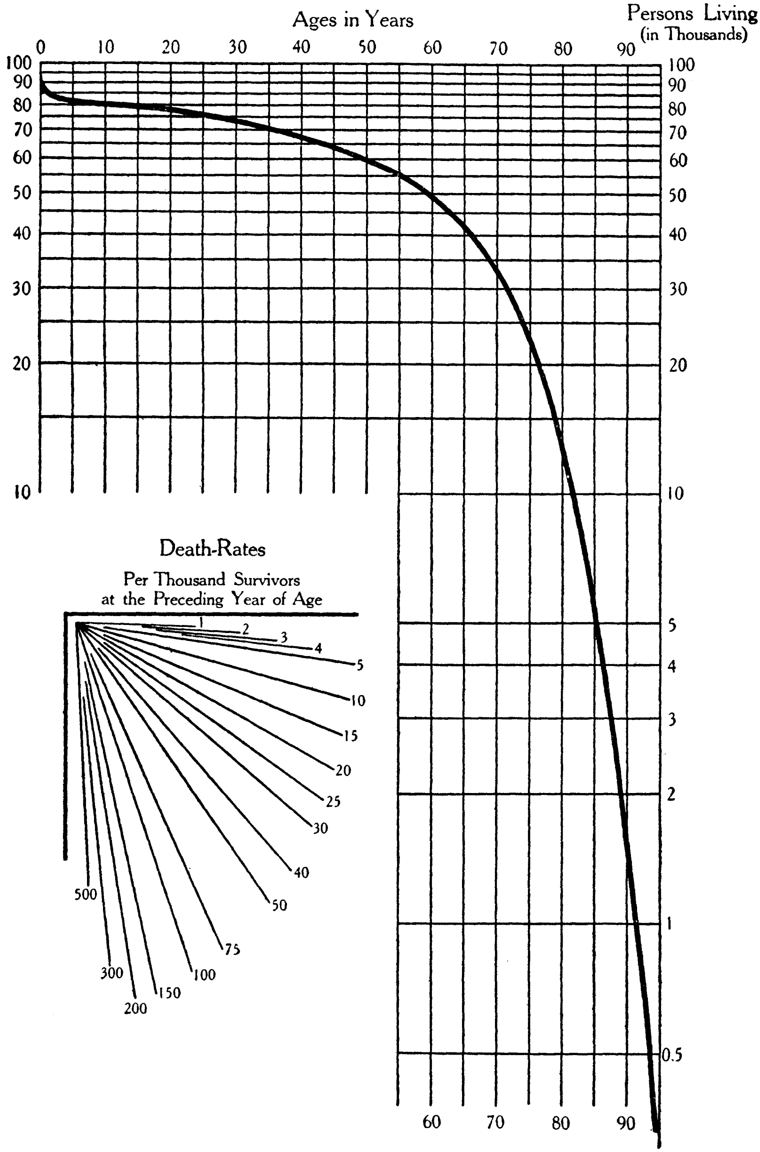


DIAGRAM IX.—MORTALITY AND SURVIVAL OF NATIVE WHITE MALES IN THE ORIGINAL REGISTRATION STATES OF THE UNITED STATES, 1910

*Logarithmic Vertical Scale*

**Data of Diagram VIII**



distance on the logarithmic scale denotes a certain proportion of change. Hence the more steeply the logarithmic curve descends, the higher is the relative mortality which it indicates. Hence, too, it is possible to provide a key to the diagram in the form of standard sample slopes and corresponding numerical death-rates, which hold true for all parts of the curve. With their aid, even in so small a figure, it can be shown that the high rate of mortality prevailing in the first year of life, taken as a whole,[1] is exceeded by the mortality of men over eighty—a fact which would remain doubtful in Diagram VIII, where the effect of the infant mortality rate is exaggerated because of the great number of individuals under consideration, and where, on account of the attenuated population at advanced ages, even the extreme mortality of the very old results in absolutely few deaths and thus in a comparatively inconspicuous descent of the curve.

Diagram IX incidentally reveals a characteristic of logarithmic diagrams which the previous figures have not brought out. The descending curve is arbitrarily cut off at age 95, leaving 330 survivors of the original 100,000. To continue the curve to age 100, thereby accounting for all but 27 survivors, would necessitate prolonging the vertical dimension of the diagram by nearly one-half, for the relative reduction of the number of survivors is rather more in the five years from 95 to 100 than in the first eighty-three years of life. To carry the curve to the point where no one survives is impossible. The logarithmic scale has no zero. Extending a logarithmic scale to zero is equivalent to reducing a finite number to zero by successive divisions, and neither task can be accomplished short of infinity. The impossibility of completing the logarithmic representation of a life-table is of identical nature with the impossibility of completely amortizing an original sum by writing off each year a fixed proportion of the remaining amount. Either process can be carried on until the residual value, or the number of survivors, as the case may be, is reduced to any assigned quantity

[1] The mortality of the first month or two is, of course, relatively much higher than the mortality of the first year as a whole. To this fact is due the marked concavity of the curve at the outset. A diagonal straight line, substituted for the curve from age 0 to age 1, would by its slope show the first-year mortality referred to in the text.

greater than zero; but neither can be continued to the point of extinction.[1]

Since logarithmic scales have no zero, logarithmic diagrams can have no base-line at zero. Indeed, they have no base-line at all; or, rather, every value of the logarithmic scale is as much a base-value as any other. This follows from the cardinal principle already repeatedly stated, that the same absolute interval stands for the same ratio of magnitudes at any and every part of a given logarithmic scale. It obviously constitutes an essential distinction between logarithmic and natural-scale diagrams. In a natural-scale diagram the importance of showing the base-line at zero of the vertical scale can hardly be urged too strongly. If this base-line be omitted, as it often is in unintelligent work, proper visual estimation of relative magnitudes is made impossible. Such omissions in complex natural-scale diagrams involving more than one base-line lead to extreme confusion and fallacy. In logarithmic diagrams fallacious effects of this particular sort are impossible; but any suggestion of a specific base-line may prove disconcerting to those unfamiliar with the logarithmic scale and may cause misconception of its character.[2]

The principles which have thus far been developed may now be recapitulated:

Throughout a given diagram, and regardless of the absolute magnitudes concerned:

(1) a given distance between any two points, measured along a logarithmic scale, indicates in every case the same ratio between the two magnitudes which the positions of the points represent;

[1] This same principle is brought out in another way by Diagram XX and the discussion on pp. 837–38.

[2] For this reason Mr. Willard C. Brinton (cf. *Graphic Methods for Presenting Facts*, p. 362) and the Joint Committee on Standards for Graphic Presentation seem ill-advised in proposing that the bottom line and the top line of every logarithmic chart should mark some power of 10 on the logarithmic scale. Mr. Bowley's authority (*Elements of Statistics*, p. 190) and the logic of the method are both against them. If for any reason it is felt to be important that the logarithmic scale should start with a power of 10, that objective may be attained, as in Diagram X, p. 825, without the necessity of a base-line. But in all logarithmic diagrams the absolute numbers of the scale are quite subordinate in importance, and should not be emphasized in such a way as to mask the essential properties of the scale as a measure of relations.

(2) when changing magnitudes are plotted to a vertical logarithmic scale, and unit intervals of time are plotted to a horizontal natural scale,

(a) the slope of a curve is always an index of the rate of relative change;

(b) a straight line represents a constant rate of relative change; and, conversely, a constant rate of relative change is always represented by a straight line;

(c) where the vertical distance between two curves is constant the variables which they respectively represent maintain always the same proportion one to the other; and, conversely, two variables constantly in the same proportion are always represented by two curves at a fixed vertical interval.

The logarithmic scale admits of no zero, and in terms of a logarithmic scale no base-line should ordinarily be indicated.

With these general principles in mind we may now consider Diagram X, in which the bank statistics of Diagram I are plotted to a logarithmic scale. The questions which Diagram I failed to answer[1] find here a ready solution, and incidentally illustrate certain useful devices for the interpretation of logarithmic diagrams in general.

The relative expansion of deposits, evidenced by the absolute rise of the upper curve in Diagram X, was plainly greater in the year following October, 1873, than in the year following October, 1907. How great it was in either year may be determined with the aid of the percentage scale of increase at the right of the main figure. This scale, it is to be noted, holds good for vertical measurements at all parts of the diagram, since its logarithmic intervals make it a scale of ratios, quite independent of absolute magnitudes. The vertical rise of the deposit curve following 1873 shows by the scale an increase of approximately 50 per cent. The rise after 1907, similarly measured, is some 38 per cent.

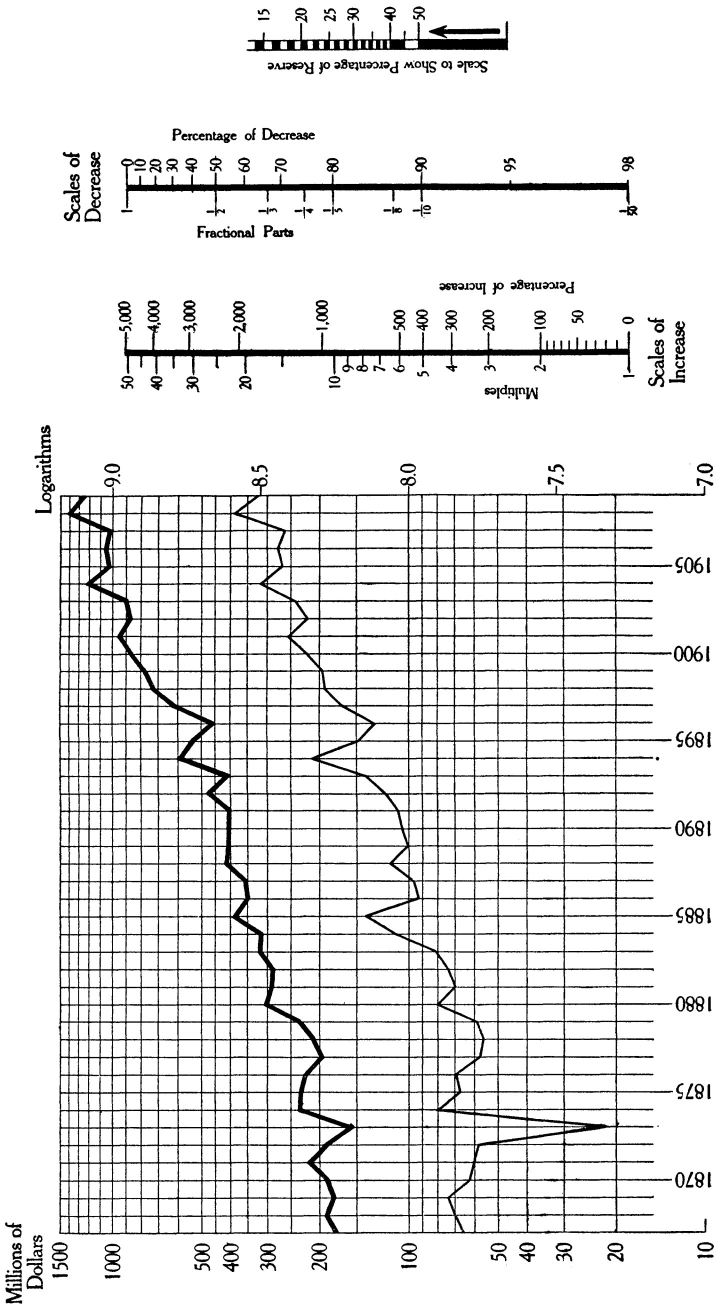
Relative decreases of deposits can be tested in a manner quite analogous by the logarithmic scale of percentage decrease. Here,

[1] Cf. p. 808, above.

DIAGRAM X.—NET DEPOSITS (HEAVY LINE) AND RESERVES (LIGHT LINE) OF THE CLEARING-HOUSE BANKS OF NEW YORK CITY, ACCORDING TO THE 41ST WEEKLY REPORT (EARLY OCTOBER) IN EACH YEAR, 1867–1909

*Logarithmic Vertical Scale*

Data of Diagram I



for convenience, the scale reads from the top downward, rather than up from the bottom, as in the scale of increase. The contraction of deposits from October, 1871, to October, 1873, as measured by the decrease scale, was about 27 per cent—appreciably greater than the contraction of some 22 per cent during the two years preceding October, 1896.

The proportion of reserve to deposits at any given date is obviously to be determined from Diagram X by measuring the appropriate vertical distance between the reserve curve and the deposit curve. For this purpose one might use the scales designed to measure increase and decrease. Thus, in October, 1905, deposits were not quite four times as great as reserves, according to the multiple scale. Interpreted by the scales of decrease, reserves were equivalent to slightly more than a quarter of the deposits, or were some 74 per cent less than the deposits. None of these statements, however, expresses reserves in the conventional way as a percentage of deposits. For convenience, therefore, a special inverse logarithmic scale is provided at the extreme right of the figure.[1] If a given vertical interval between the reserve curve and the deposit curve is laid off on this scale, from the bottom upward, the reading of the inverse scale states the reserve directly as a percentage of deposits. In October, 1905, it thus appears that the reserve stood at 26 per cent. The rough parallelism of the two curves throughout their whole course shows that the percentage of reserve has not greatly changed. Nevertheless, it is tolerably clear that the reserves held in early October were rather larger before 1870 than since 1895; for in the former period the curves are nearer together. The last of the questions which Diagram I left unsettled thus finds its answer in Diagram X.

The inverse percentage scale introduced in Diagram X is so serviceable a device as to justify further illustrations of its use. In Diagram XI it permits reading dutiable merchandise imports as a percentage of total imports of merchandise; and, when employed to measure the vertical distance between the customs curve and one of the curves of imports, presents directly the average ad valorem rate of duty for any year, reckoned, as the case may be, on

[1] This scale is, of course, simply a scale of the reciprocals of the numbers which would appear at the corresponding points on the logarithmic scale of multiples.

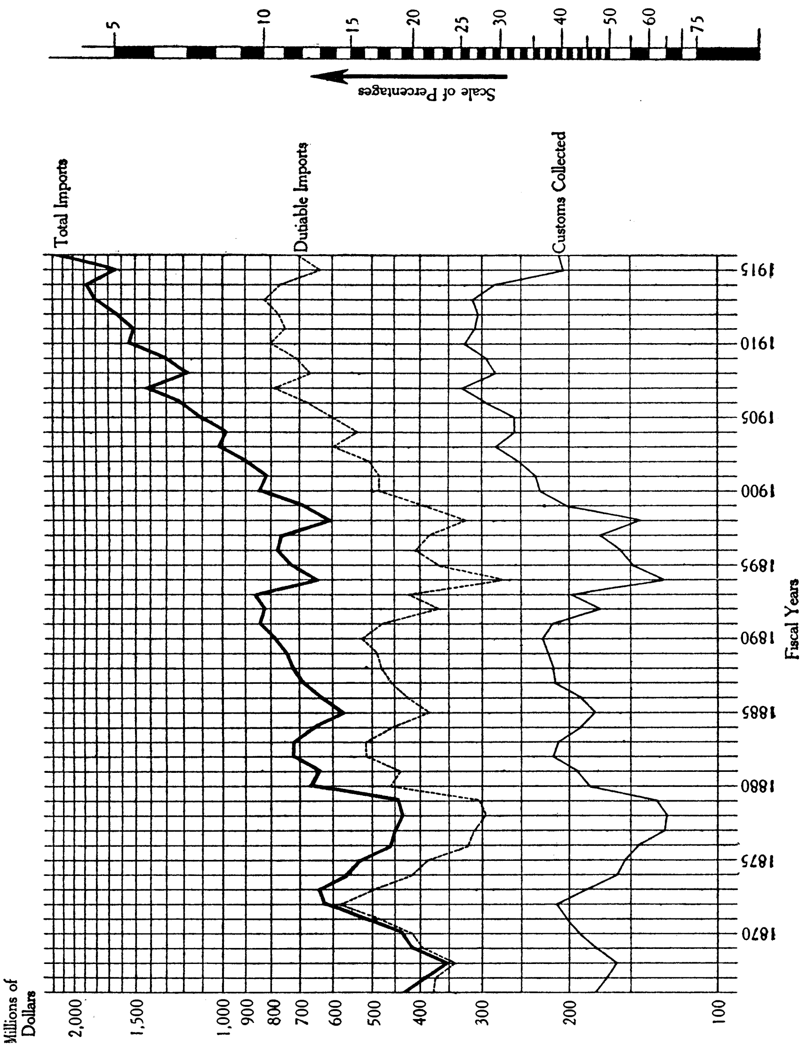



DIAGRAM XI.—TOTAL MERCHANDISE IMPORTS, DUTIABLE MERCHANDISE IMPORTS, AND CUSTOMS COLLECTED, FOR THE UNITED STATES, IN EACH FISCAL YEAR, 1866–1916

*Logarithmic Vertical Scale*

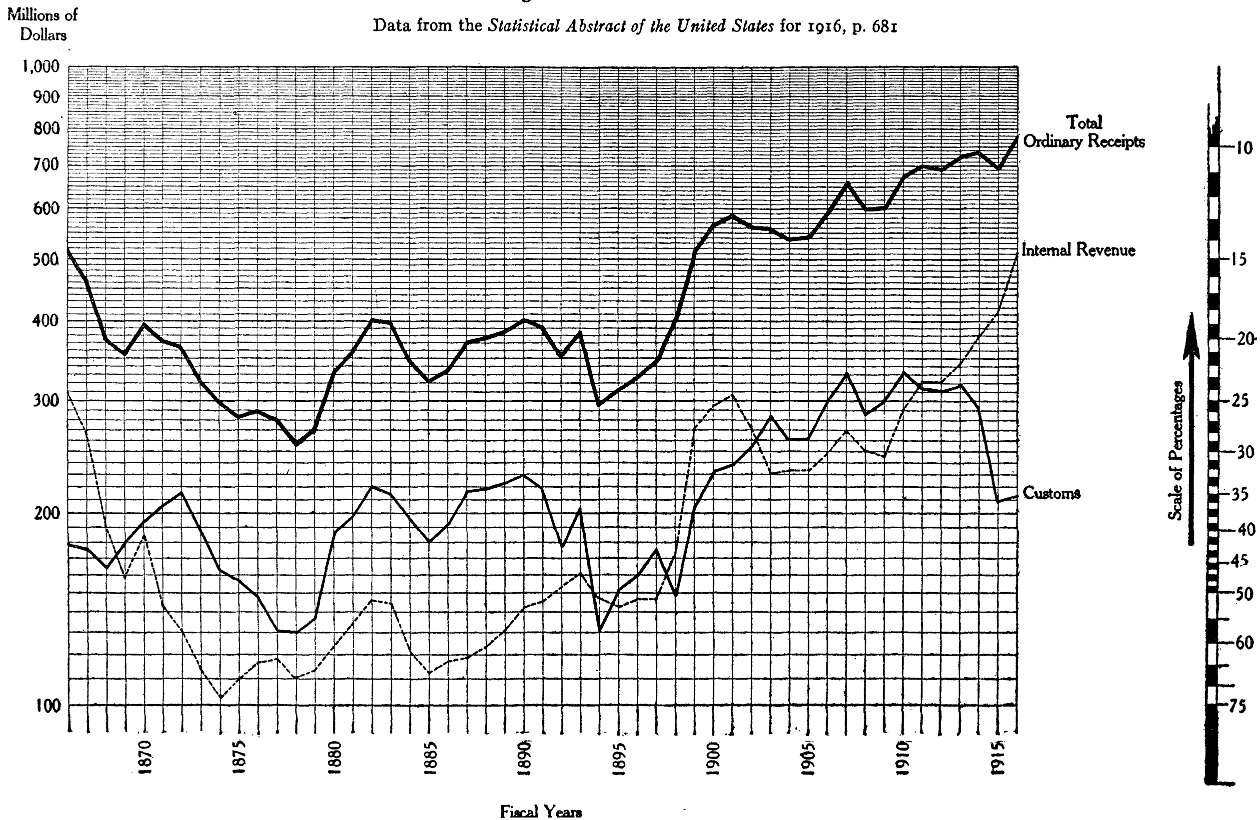
Data from the *Statistical Abstract of the United States* for 1916, p. 683

imports as a whole or on dutiable imports only. In Diagram XII it serves to translate into percentages of total ordinary revenue the annual receipts from customs and from internal revenues, respectively. Diagram XIII is a more extreme example of its possibilities. Here the scale, in percentages or in the per-thousand equivalents, not only interprets as a general death-rate the interval

DIAGRAM XII.—TOTAL ORDINARY RECEIPTS OF THE UNITED STATES TREASURY IN EACH FISCAL YEAR, 1866–1916, WITH THE RECEIPTS FROM INTERNAL REVENUE AND FROM CUSTOMS (INCLUDING THE TONNAGE TAX)

*Logarithmic Vertical Scale*

Data from the *Statistical Abstract of the United States* for 1916, p. 681



between the total deaths curve and the curve of population, but makes possible, by analogous measurements, the determination of death-rates for each specified disease, and the expression of the deaths from each disease as a percentage of all deaths. The noteworthy feature of this illustration is the extreme range of magnitudes in the data presented. The estimated number of persons in 1914 is nearly two thousand times as great as the number of deaths from cancer in 1900. The inclusion of such diverse numbers in one diagram has necessitated a rather severe compression of the vertical scale, with the result that in a diagram of convenient breadth the curves are flatter and less vigorous than would be desirable. But, if the logarithmic method is taxed by the demands of this problem, the natural-scale method fails outright, as is revealed by Diagram XIV. The population curve is here clear enough;[1] but the curve of total deaths is abased into almost featureless insignificance. As for the curves of deaths from special diseases, they never emerge into visibility, but lie blurred in the base-line. The highest point on the highest of these curves would be but 0.006 of an inch above the zero of the scale.

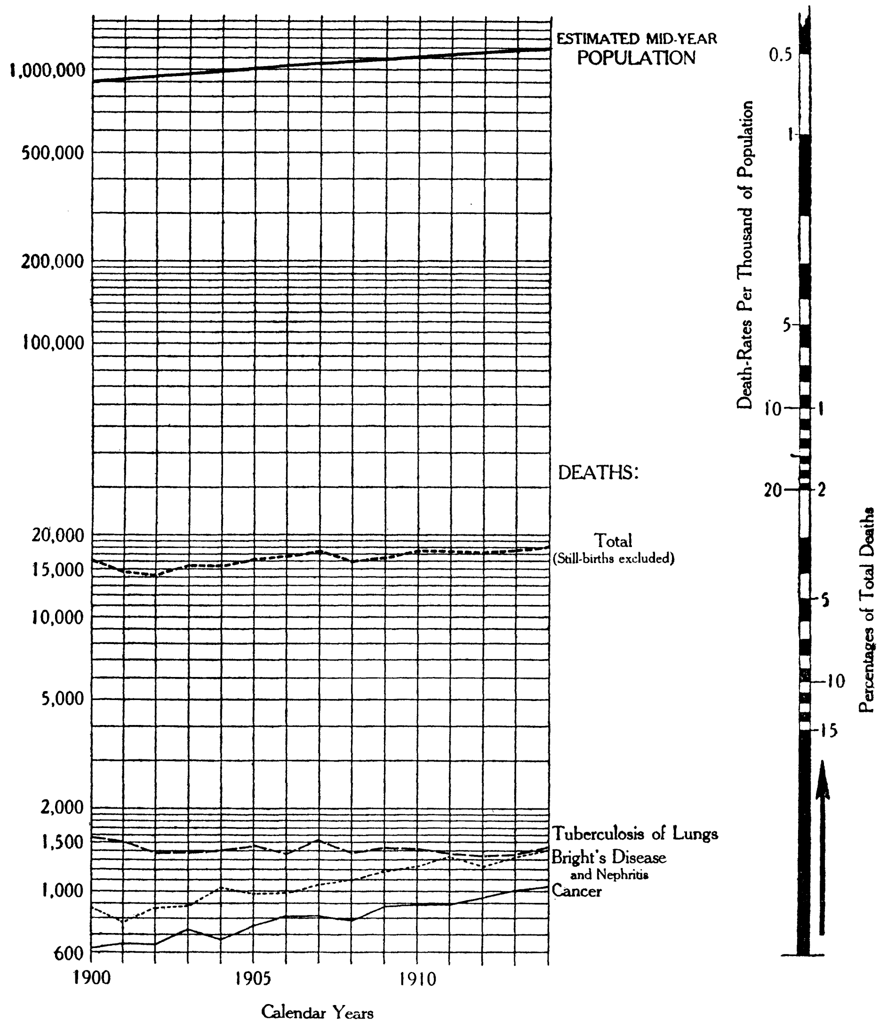
Thus another merit of no slight importance is to be recorded for the logarithmic scale: it is far superior to the natural scale for effecting comparisons when very small and very large quantities must be taken into account concurrently. Nor is this superiority manifest only in such figures as Diagram XIII. Whenever a historical curve records extreme growth, the same advantage is found. It is not necessary to dwarf the small beginnings in order to keep the later development within manageable dimensions. A study of Diagrams III and IV will illustrate this point. More striking illustration is offered in Diagrams XV and XVI. The production of tinplate in 1891 and the years immediately following was so small that the ordinary diagram (Diagram XV) leaves inconspicuous the extremely rapid rate of progress in output during those first years. The logarithmic diagram (Diagram XVI) quite reverses the emphasis. Plainly, the recent increase has been far from proportionate to the exuberant growth of the infant industry.

[1] It gives, in fact, interesting proof of the basis upon which the Bureau of the Census estimates the population in intercensal years. The curve is a straight line. Plainly, therefore, the population is assumed to have increased annually since 1900 by one-tenth of the total increase for the intercensal decade 1900–1910.

DIAGRAM XIII.—ESTIMATED POPULATION, TOTAL DEATHS, AND DEATHS FROM CERTAIN SPECIFIED DISEASES, IN CONNECTICUT, FOR EACH YEAR, 1900–1914

*Logarithmic Vertical Scale*

Data from Mortality Reports of the United States Bureau of the Census

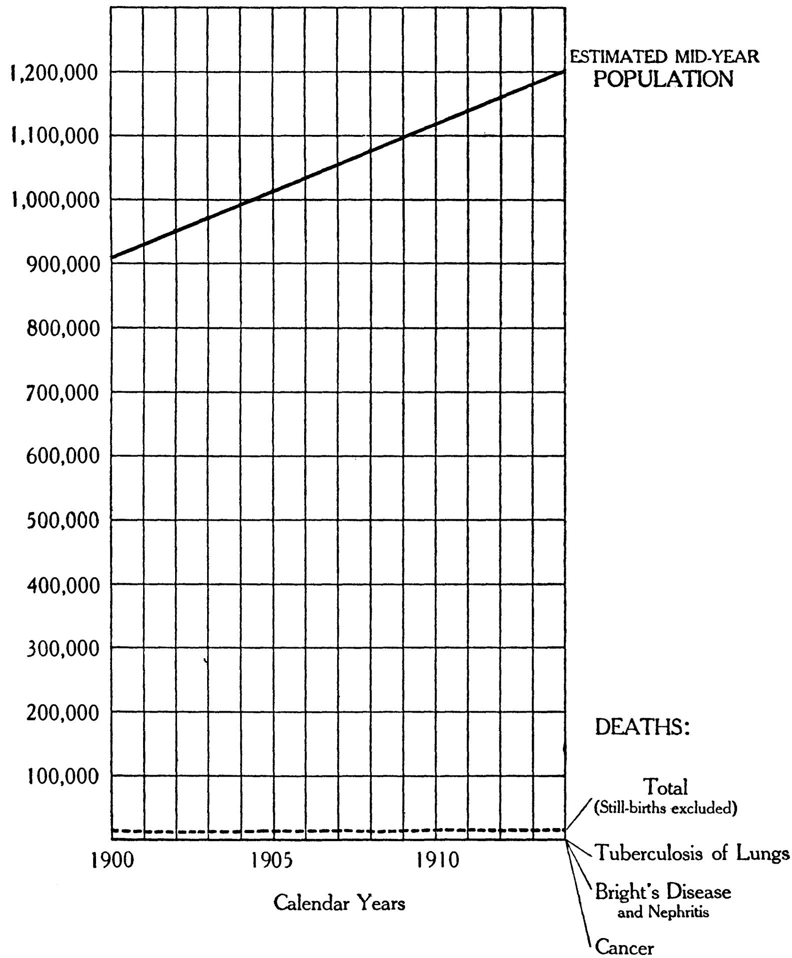


Although the years of small beginnings in a historical record may present no features that require special consideration, the logarith-

DIAGRAM XIV.—ESTIMATED POPULATION, TOTAL DEATHS, AND DEATHS FROM CERTAIN SPECIFIED DISEASES, IN CONNECTICUT, FOR EACH YEAR, 1900–1914

*Natural Scale*

Data of Diagram XIII



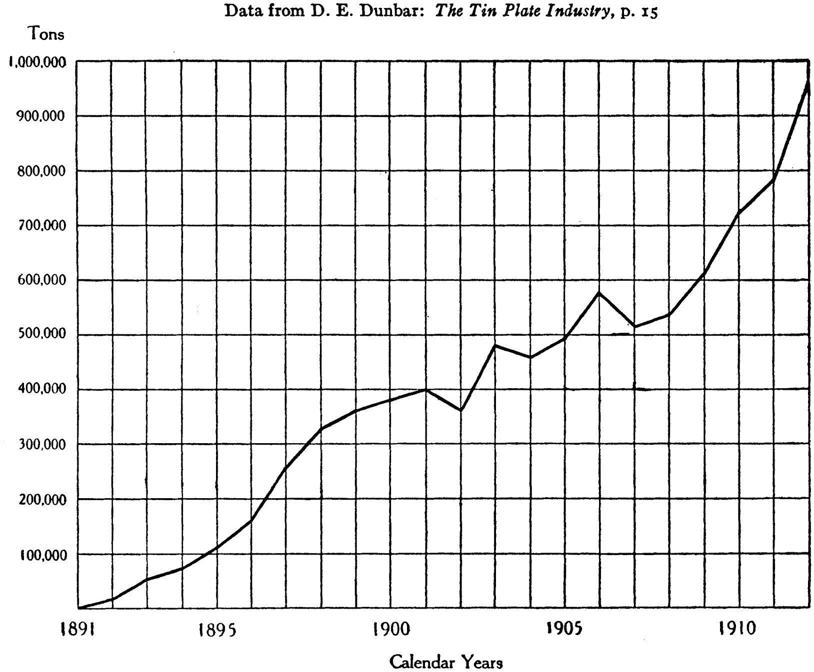

mic historical diagram is again advantageous whenever substantially the same rate of relative increase characterizes the whole period

under review. In such cases the general trend or growth-axis of the logarithmic curve will of course be nearly straight. This is interesting for its evidence of consistent growth. It has the further technical merit of permitting the trend of the curve to be approximately maintained throughout at any desired slope by the mere choice of dimensions for the diagram. Hence such curves can

DIAGRAM XV.—ANNUAL PRODUCTION OF TINPLATE IN THE UNITED STATES, 1891–1912

*Natural Scale*

Data from D. E. Dunbar: *The Tin Plate Industry*, p. 15




readily be kept close to an inclination of 45°, with the result that irregularities of direction are much more easily noticed than if the slope were as steep or as flat as in natural-scale diagrams some parts of the curve often must be.

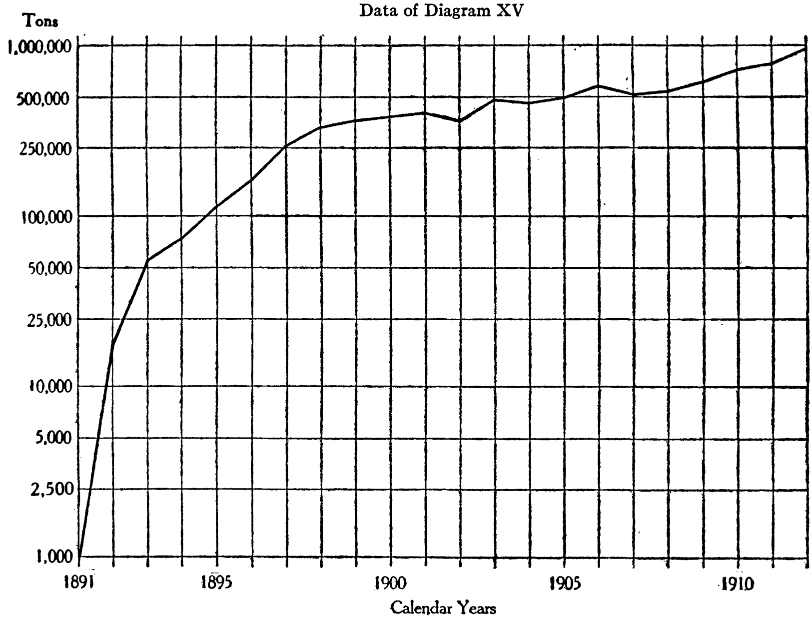
For the plotting of index-numbers logarithmic diagrams are particularly appropriate, for here the numbers themselves are ratios, and their relative aspect is important. If an index number of general prices should rise from 80 to 100, and later from 100 to

120, the two changes would appear of equal significance in an ordinary diagram. Yet the first is an increase of 25 per cent, the second, an increase of but 20 per cent. In their effects upon the purchasing power of stated money incomes the two changes are by no means the same. A logarithmic diagram reveals their significant difference. Diagrams XVII and XVIII contrast the natural-scale method with the logarithmic-scale method in the case of the general

DIAGRAM XVI.—ANNUAL PRODUCTION OF TINPLATE IN THE UNITED STATES, 1891–1912

*Logarithmic Vertical Scale*

Data of Diagram XV




index number of wholesale prices from 1890 to 1914, published by the United States Bureau of Labor Statistics. It will be remarked that the logarithmic figure, which does not require a zero base-line in order to convey a true sense of relative values, permits a considerable saving of space.

The foregoing discussion has referred throughout to graphic constructions in which logarithmic intervals may be substituted for uniform intervals in the vertical scale. Such constructions include

DIAGRAM XVII.—COURSE OF THE GENERAL INDEX NUMBER OF WHOLESALE PRICES PUBLISHED BY THE UNITED STATES BUREAU OF LABOR STATISTICS, 1890–1914

(AVERAGE PRICES FOR THE PERIOD 1890–99 ARE TAKEN AS 100)

*Natural Scale*

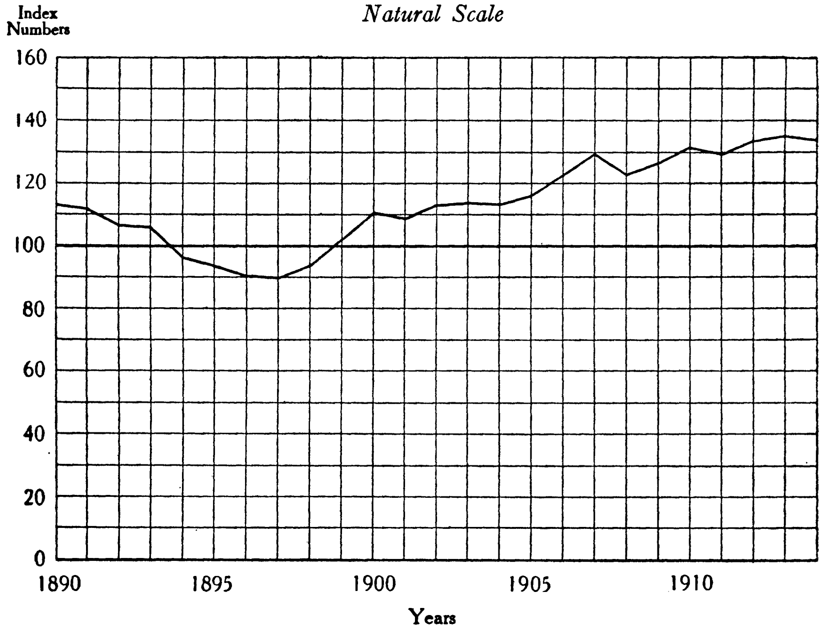
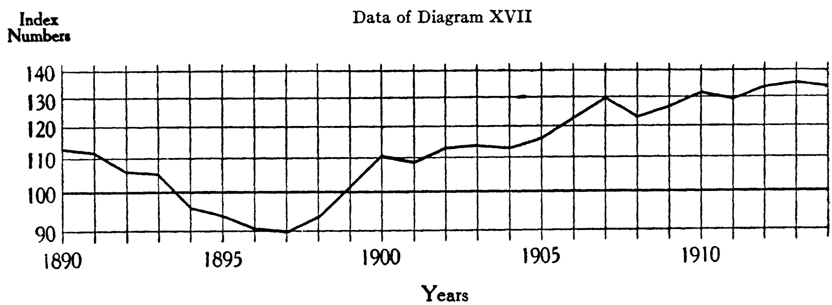



DIAGRAM XVIII.—COURSE OF THE GENERAL INDEX NUMBER OF WHOLESALE PRICES PUBLISHED BY THE UNITED STATES BUREAU OF LABOR STATISTICS, 1890–1914

*Logarithmic Vertical Scale*

Data of Diagram XVII



nearly all the important uses to which logarithmic scales have been put in statistical representation. The reason for the prevalence of this usage is sound, but it admits of exceptions. On occasion the horizontal scale may be selected for logarithmic subdivision.

The vertical scale, by convention, is the scale of the "dependent variable," while the "independent variable" is plotted on the horizontal scale. That is, the horizontal scale defines the questions to be answered: the vertical scale shows the answer. The statistical investigator determines the independent variable when he decides upon the terms of his problem. His study of imports, for example, is to be a study by fiscal years. From the moment of that decision, time, in fiscal years, is the independent variable in his investigation. When his results are presented in graphic form, the years will appear along the horizontal scale. Persons who examine his diagrams will recognize the years as the basis of the study, but will fix their immediate attention upon the dependent variable—the annual amounts of imports, measured upon the vertical scale. Because the values of the dependent variable are thus the matter of immediate interest in a diagram, the statistician not unnaturally devotes to them his most careful expository devices, and among others the device of the logarithmic scale for the special elucidation of relative magnitudes.

But precisely because of this emphasis upon the dependent variable it is unfortunate to becloud it with any uncertainty which, for the uninitiated, may lurk in the logarithmic intervals. Since, on the other hand, values of the independent variable are so much taken for granted—since, for example, the notion of successive years of observation is so simple and familiar—it might seem expedient, if it were possible, to use the ratio-scale for the independent variable, to the end that there should emerge a diagram combining the advantages of the logarithmic method with the simplicity of a natural vertical scale. This objective, though not wholly realizable, may be approached in special and limited instances.

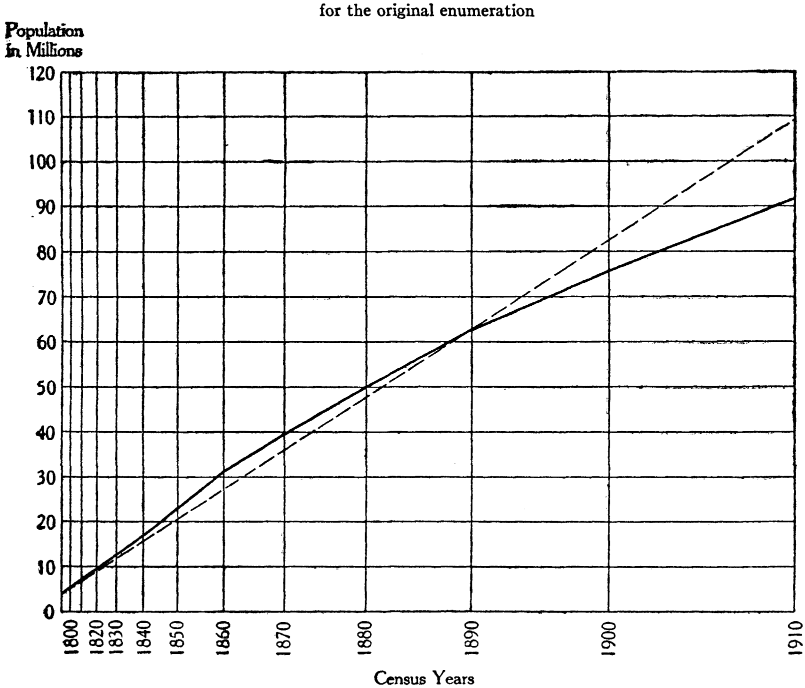
Diagram XIX illustrates a simple attempt of the sort. It is designed to contrast the actual growth of population in the United States with the growth that would have taken place if a constant geometrical rate of increase had doubled the numbers every twenty-five years since 1790. The vertical scale, recording population in

millions, has the natural, uniform intervals. The dates of the successive censuses are arbitrarily marked off on the horizontal scale at such points that the broken-line curve of the figure, representing the hypothetical doubling of population every twenty-five

DIAGRAM XIX.—ACTUAL GROWTH OF THE POPULATION OF THE UNITED STATES (CONTINUOUS LINE) CONTRASTED WITH A HYPOTHETICAL DOUBLING EVERY 25 YEARS SINCE 1790 (BROKEN LINE)

*Geometric Horizontal Scale*

Data from 13th Census of the United States, I, 24. The revised estimate for 1870 has been substituted for the original enumeration




years, shall be a straight line. In other words, the decennial intervals of the horizontal scale are made proportional to the assumed absolute increases of population during the corresponding periods, and therefore constitute a geometrical series. When the actual growth of population, according to census enumerations,[1] is plotted

[1]As before, in Diagrams V and VI, the revised estimate for 1870 has been substituted for the original enumeration.

to these scales, the course of the resulting curve, viewed in relation to the straight line of the hypothesis, effectively shows to what extent the predicted rate of doubling has in fact been realized. The deviations of the actual curve (the heavy, continuous curve in Diagram XIX) from the hypothetical straight line are in this construction easily read from a natural scale; but they are of course depicted as absolute and not as relative deviations.

In Diagram XX a similar horizontal scale is utilized for charting depreciation on the basis of an annual shrinkage in value equal to 10 per cent of the value at the beginning of the year. Each interval on the time-scale is taken as $\frac{9}{10}$ of the preceding interval; and consequently the required depreciation curves are straight lines, whatever the original values subject to depreciation.

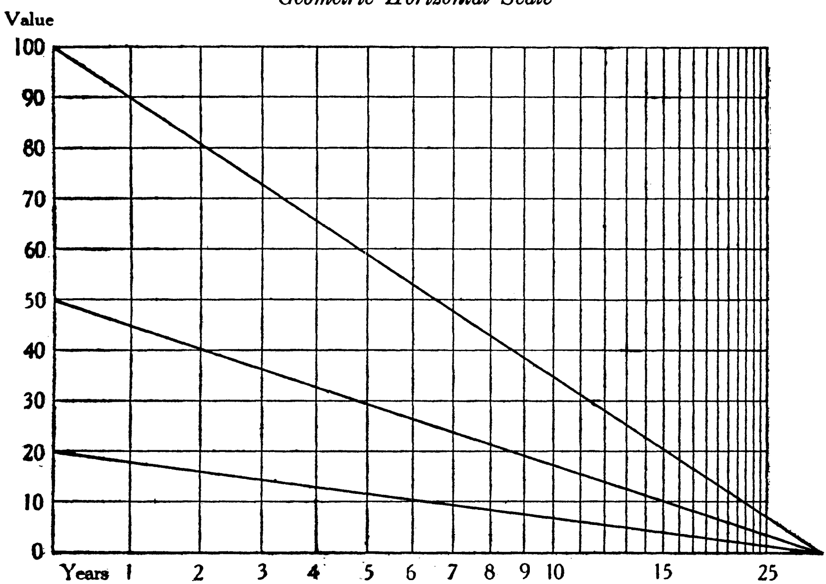
For its special purpose the construction of Diagram XX is simple and convenient. The method there employed is, however, of much more restricted use than the method adopted in the somewhat analogous case of Diagram VII. In that figure (which might, of course, be read backward to determine depreciation instead of compounded increase) the several curves that show increase at 6 per cent compound interest are not merely straight lines, but lines of the same characteristic slope, whatever the initial amount invested. In Diagram XX the different depreciation curves are not parallel: the slope of each depends on the initial value. Again, whereas in Diagram VII compound increase at any rate other than 6 per cent would be denoted by a straight line of another and equally characteristic slope, curves drawn in Diagram XX to show depreciation at a rate other than 10 per cent per year would not be straight lines. The device of a geometric horizontal scale, as applied in Diagrams XIX and XX, can be adapted to give straight-line curves for any one rate of geometric increase or decrease, but for only one.

The construction of Diagram XX seems at first glance to involve the paradox of complete amortization by the repeated writing off of a fixed annual percentage. The vertical scale of values, being a natural scale, has of course a definite zero base-line which the depreciation curves, being oblique straight lines, must cut if they are sufficiently prolonged. They do in fact cut it in a common

point, as the figure demonstrates. But this point represents, on the scale of years, an infinity of time. Though the graduation of the horizontal scale be extended by any finite number of the dwindling one-year intervals, this point will not be quite reached. Thus the impossibility of final amortization which in a depreciation chart of the type of Diagram IX would be attested by the infinite

DIAGRAM XX.—DEPRECIATION CHART, SHOWING THE RESIDUAL VALUE, AFTER ANY SPECIFIED NUMBER OF YEARS, OF A GIVEN ASSET DEPRECIATING ANNUALLY BY 10 PER CENT OF ITS VALUE AT THE BEGINNING OF THE YEAR

*Geometric Horizontal Scale*




remoteness of the zero value on a logarithmic vertical scale, here manifests itself in the infinitesimal effect of added years.

Of diagrams with both scales divided logarithmically a single example will suffice. Diagram XXI exhibits, on a double logarithmic scale, the numbers of incomes in the United States which, according to the personal income-tax returns for the calendar year 1916, equaled or exceeded the annual amounts specified. Diagram XXII, with natural scales, presents the same data for amounts up to \$100,000. To have presented the full range of the

data in this form and within the limits of the page would have compelled plotting the amounts of income on about one-tenth the scale actually adopted, with the result of rendering virtually invisible

DIAGRAM XXI.—DISTRIBUTION OF PERSONAL INCOMES SUBJECT TO THE UNITED STATES FEDERAL INCOME TAX, AS REPORTED FOR THE CALENDAR YEAR 1916

The curve shows the total number of incomes equal to or exceeding each specified annual amount

*Double Logarithmic Scale*

Data from the *Statistical Abstract of the United States* for 1916, pp. 648–49

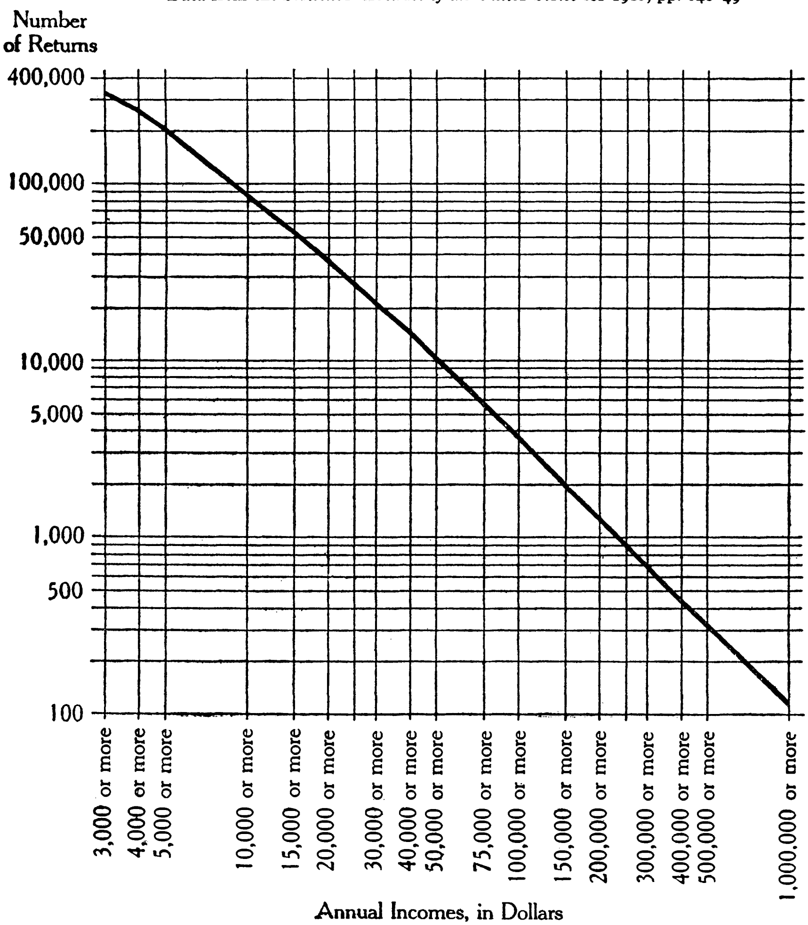
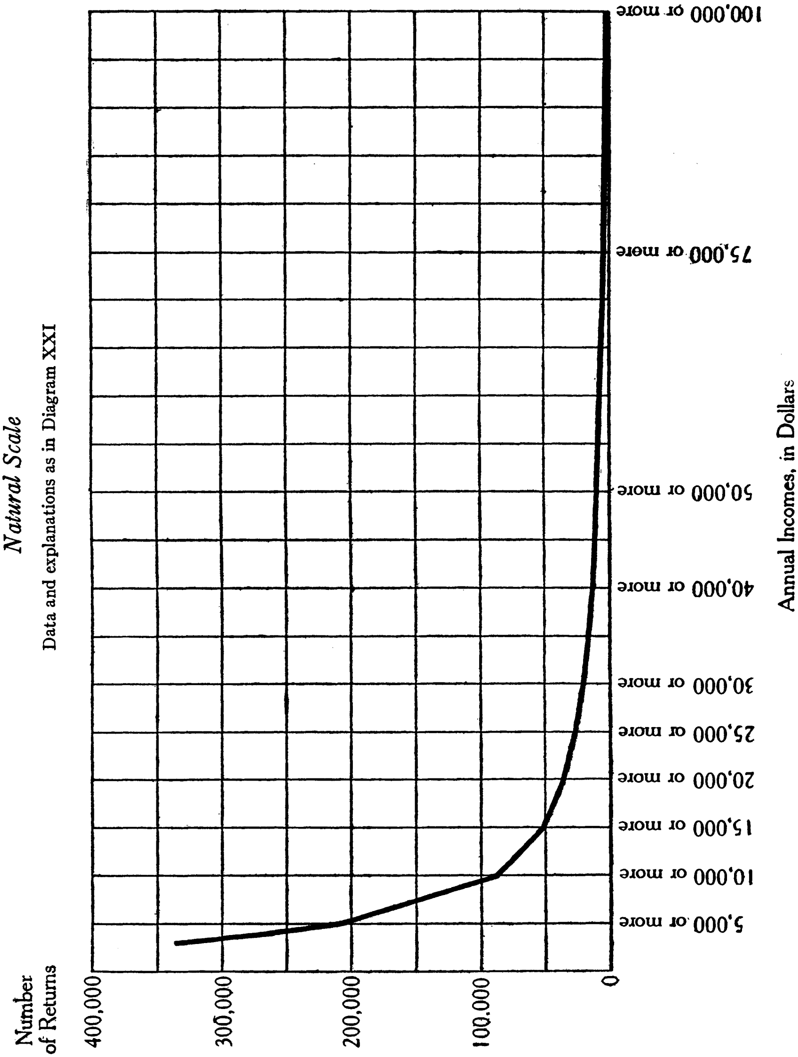


DIAGRAM XXII.—DISTRIBUTION OF PERSONAL INCOMES SUBJECT TO THE UNITED STATES FEDERAL INCOME TAX, AS REPORTED FOR THE CALENDAR YEAR 1916

*Natural Scale*

Data and explanations as in Diagram XXI



the detailed distribution of incomes under $10,000 a year—that is, of nearly three-fourths of all the incomes reported. Diagram XXI thus proves again the economy of space which logarithmic scales may effect. In this particular case it serves also a more interesting purpose; for it tests, by these special data, Professor Pareto's generalization, according to which the curve of the distribution of incomes, plotted in this manner to a double logarithmic scale, approximates a straight line.

If it were the aim of this article to explore the full possibilities of logarithmic diagrams as a means of statistical analysis, many other interesting instances might no doubt be adduced. But the intention, as was announced at the outset, is rather to exemplify, and thus to familiarize, the more obvious advantages of the logarithmic scale in graphic representation. That intention has perhaps now been in some measure accomplished.

From the illustrations which have been offered it will have appeared first of all that logarithmic diagrams present ratios and relative changes as directly and simply (though not, to the uninitiated eye, so obviously) as natural-scale diagrams present absolute differences. Consequently the logarithmic method is peculiarly effective when the data are essentially relative; when they exhibit a tendency to increase or decrease at a fixed relative rate; or when significant proportionalities between different series of data are to be demonstrated. Incidentally it serves to economize space, and thus permits the inclusion of very diverse magnitudes in the same figure. These are real advantages, which clearly justify the use of logarithmic constructions in a considerable range of graphic work—sometimes by themselves, sometimes in conjunction with other forms of representation. How extensively such constructions will or should supplant ordinary figures on the natural scale need not now be argued. It is enough to make known their fundamental properties. When these are generally appreciated, we may trust the ingenuity and judgment of statisticians to find for logarithmic diagrams the place that they deserve.

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